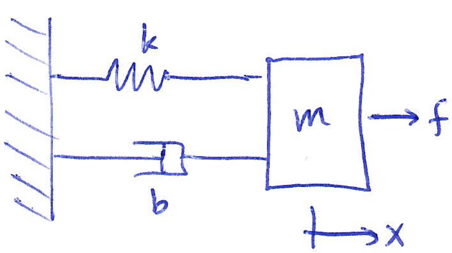
ME 4555 - Lecture 19 - Second order Systems

A second-order system has two poles. We are interested in the case where there is oscillation, but we will look at all cases. All stable 2nd order systems can be described by 3 numbers: Gain (K), natural frequency (ω_n, pronounced "omega-n"), and damping ratio (ζ, pronounced "zeta").

The general canonical form is:

$$\boxed{G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Ex spring-mass-damper.



Eqn of motion:

$$m\ddot{x} + b\dot{x} + kx = f.$$

Transfer function: $$\frac{X}{F} = \frac{1}{ms^2 + bs + k}$$

Rearrange: $$\frac{X}{F} = \frac{(1/m)}{s^2 + (b/m)s + (k/m)}$$

$$= \frac{(1/k)(k/m)}{s^2 + 2\left(\frac{b}{2\sqrt{km}}\right)\left(\frac{\sqrt{k}}{\sqrt{m}}\right)s + \frac{k}{m}}$$

so: $$\begin{cases} K = \frac{1}{k} \\ \omega_n = \sqrt{\frac{k}{m}} \\ \zeta = \frac{b}{2}\sqrt{\frac{1}{km}} \end{cases}$$

There are several cases to consider, depending on whether we have real roots, complex roots, repeated roots. Which case we belong to depends only on the value of ζ (not on K or ω_n).

The roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$ are (quadratic formula):

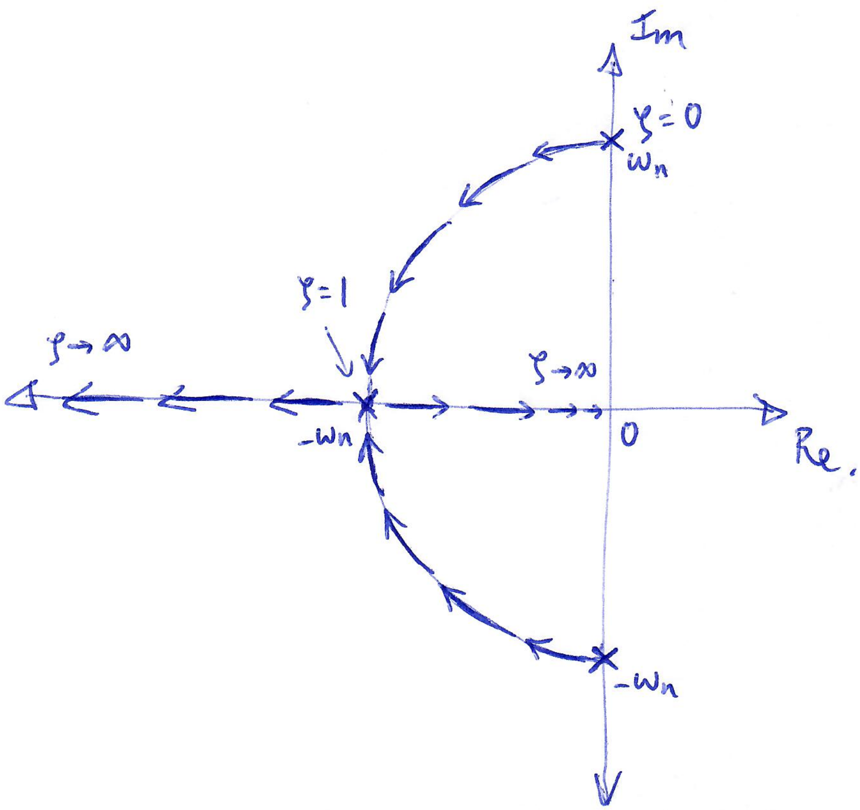
$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

If $0 \le \zeta < 1$, we have $s = -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2}$ (complex conj. pair)

If $\zeta = 1$, we have $s = -\zeta\omega_n = -\omega_n$ (repeated real roots)

If $\zeta > 1$, we have $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$ (distinct negative real roots)

As we vary ζ from 0 to ∞, the roots travel from the ~~y~~ axis to the x-axis and split apart. (circular path!)



ζ	pole locations
0	$(\pm i)\,\omega_n$
0.3	$(-0.3 \pm 0.954i)\,\omega_n$
0.5	$(-0.5 \pm 0.866i)\,\omega_n$
0.8	$(-0.8 \pm 0.6i)\,\omega_n$
1	$\{-1, -1\}\,\omega_n$
1.5	$\{-2.62, -0.382\}\,\omega_n$
5	$\{-9.9, -0.1\}\,\omega_n$
20	$\{-40, -0.025\}\,\omega_n$
∞	$\{-\infty, 0\}\,\omega_n$

The case $\zeta > 1$ is called overdamped. In this scenario, the damping effect is so large that there is no possibility of oscillation. The system is sluggish. Think of a spring-mass-damper placed in molasses. When $\zeta > 1$, we have two distinct real roots, both stable. Example: the DC motor from lecture 18.

The case $\zeta = 1$ is called critically damped. We have repeated real roots, both stable.

The case $0 < \zeta < 1$ is called underdamped. The system will overshoot its goal before settling down, possibly with many oscillations. Smaller ζ means more oscillations. The case $\zeta = 0$ is called undamped

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Both overdamped and critically damped cases have exponential responses; no sinusoids.

overdamped ($\zeta > 1$): poles: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$\Rightarrow$ impulse response: $r_1 e^{s_1 t} + r_2 e^{s_2 t}$

critically damped ($\zeta = 1$): poles: $s_{1,2} = -\omega_n$

$\Rightarrow$ impulse response: $r_1 e^{-\omega_n t} + r_2 t e^{-\omega_n t}$

repeated pole!

Step response – underdamped case $(0 < \zeta < 1)$

$$Y(s) = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K}{s} - \frac{Ks + 2K\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Complete the square in the denominator: $s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 + \underbrace{(1-\zeta^2)\omega_n^2}_{\omega_d^2}$

Define: $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}\text{, the "damped frequency".}}$

separate transfer function and find $\mathcal{L}^{-1}$.

$$Y(s) = K\left[\frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}\right]$$

$$= K\left[\frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}\right]$$

$$= K\left[\frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}\right]$$

$\mathcal{L}^{-1}$

$$y(t) = K\left[1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)\right]$$

$\boxed{\text{Define } \phi = \arccos(\zeta). \text{ So } \cos\phi = \zeta \text{ and } \sin\phi = \sqrt{1-\zeta^2}.}$

$$\Rightarrow y(t) = K\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left(\sin\phi\cos(\omega_d t) + \cos\phi\sin(\omega_d t)\right)\right].$$

$$\boxed{y(t) = K\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\right]}$$

Step response (cont'd)

$$y(t) = K\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)\right]$$

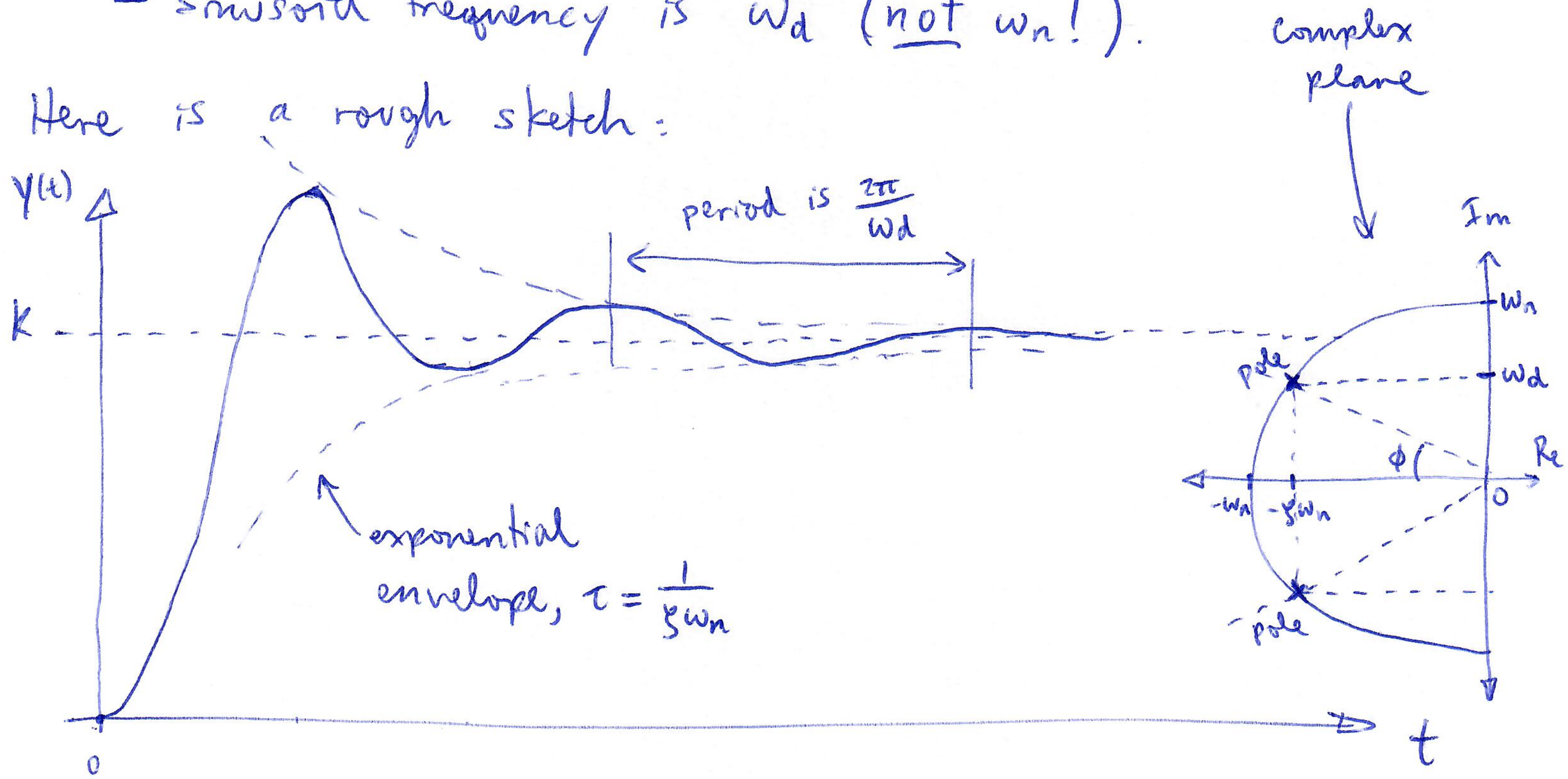
Observations: - As $t \to \infty$, $y(t) \to K$. (steady-state value).

- exponential decay governed by $\zeta\omega_n$. Larger means faster decay. In fact, we already are familiar with $e^{-t/\tau}$. So to understand $e^{-\zeta\omega_n t}$, think: $\boxed{\tau = \frac{1}{\zeta\omega_n}}$

  this is the time constant for the decay part.

- sinusoid frequency is ω_d (<u>not</u> ω_n!).

Here is a rough sketch:



There are many other useful characteristics: how much is overshoot? what is the settling time? etc. we'll see more next lecture!